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Gaussian quadratures for oscillatory integrands☆

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Abstract

We consider a Gaussian type quadrature rule for some classes of integrands involving highly oscillatory functions of the form $f(x) = f_1(x)\sin\zeta x + f_2(x)\cos\zeta x$, where $f_1(x)$ and $f_2(x)$ are smooth, $\zeta \in \mathbb{R}$. We find weights σ_ν and nodes x_ν, $\nu = 1, 2, \ldots, n$, in a quadrature formula of the form $\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{\nu=1}^{n} \sigma_\nu f(x_\nu)$ such that it is exact for all polynomials $f_1(x)$ and $f_2(x)$ from $\mathcal{P}_{n-1}$. We solve the existence question, partially.

Keywords: Orthogonality; Gaussian quadratures; Highly oscillatory integrand; Exponential-fitting Gaussian quadratures

1. Introduction

In this work we focus on an idea of using the exponential fitting considered by Ixaru and Paternoster (see [1,2]), namely, we consider the following quadrature formula:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{k=1}^{n} \sigma_k f(x_k) + R_n(f), \tag{1.1}$$

where the nodes x_k and the weights σ_k, $k = 1, \ldots, n$, are chosen such that this quadrature formula is exact on the linear span $\mathcal{F}_{2n}(\zeta)$ of the following functions: $x^k\cos\zeta x, x^k\sin\zeta x$, $k = 0, 1, \ldots, n-1$, $\zeta \in \mathbb{R}$. Notice that for $\zeta \neq 0$ we have $\dim \mathcal{F}_{2n}(\zeta) = 2n$. Also, we mention that it is enough to consider only the case $\zeta > 0$, because $\mathcal{F}_{2n}(-\zeta) = \mathcal{F}_{2n}(\zeta)$. The case $\zeta = 0$ is trivial, since $\mathcal{F}_{2n}(0)$ reduces to a pure polynomial set, i.e., $\mathcal{F}_{2n}(0) = \mathcal{P}_{n-1}$ (the set of algebraic polynomials of degree at most $n-1$).

The existence question for the quadrature rule (1.1) is not solved, yet. In this work we solve the existence question, partially, with the solution presented in Theorem 2.10.

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2. Main result

For a given $n \in \mathbb{N}$ and the set of nodes $\{x_1, \ldots, x_n\}$ we put $\mathbf{x} = (x_1, \ldots, x_n)$ and define the *node polynomial* $\omega(x) = \omega^{(n)}(x)$ by $\omega(x) = \prod_{k=1}^{n}(x - x_k)$. For brevity we introduce, for $\nu, \mu = 1, \ldots, n$, the following notation:

$$\omega_\nu(x) = \frac{\omega(x)}{x - x_\nu} = \prod_{k \neq \nu}(x - x_k), \qquad \omega_{\nu,\mu}(x) = \frac{\omega(x)}{(x - x_\nu)(x - x_\mu)} = \prod_{k \neq \nu,\mu}(x - x_k),$$

and $\ell_\nu(x) = \omega_\nu(x)/\omega_\nu(x_\nu)$, as well as

$$\Phi_\nu(\mathbf{x}) = \int_{-1}^{1} \omega_\nu(x) \sin \zeta(x - x_\nu)\, \mathrm{d}x, \quad \nu = 1, \ldots, n. \tag{2.1}$$

First, we give the explicit solution for the weights in the quadrature rule (1.1).

Theorem 2.1. *Suppose we are given mutually different nodes x_k, $k = 1, \ldots, n$, of the quadrature rule (1.1). Then the weights can be expressed in the following form:*

$$\sigma_k = \int_{-1}^{1} \ell_k(x) \cos \zeta(x - x_k)\, \mathrm{d}x, \quad k = 1, \ldots, n. \tag{2.2}$$

Proof. Since $\omega_\nu(x) \in \mathcal{P}_{n-1}$ and $\omega_\nu(x) \cos \zeta(x - x_\nu) = \omega_\nu(x) \cos \zeta x_\nu \cos \zeta x + \omega_\nu(x) \sin \zeta x_\nu \sin \zeta x$, $\nu = 1, \ldots, n$, we conclude that $\omega_\nu(x) \cos \zeta(x - x_\nu) \in \mathcal{F}_{2n}(\zeta)$, $\nu = 1, \ldots, n$. For these functions the quadrature formula (1.1) is exact, and therefore, for $\nu = 1, \ldots, n$, we have

$$\int_{-1}^{1} \omega_\nu(x) \cos \zeta(x - x_\nu)\, \mathrm{d}x = \sum_{k=1}^{n} \sigma_k \omega_\nu(x_k) \cos \zeta(x_k - x_\nu) = \sigma_\nu \omega_\nu(x_\nu),$$

i.e., (2.2). □

This theorem also implies the uniqueness of the weights once nodes are given. An immediate consequence of this theorem is that we can consider the weights as continuous functions of nodes on any closed subset of $\mathbb{R}^n$ which does not contain points with some pair of the same coordinates. This is a consequence of Lebesgue theorem of dominated convergence (cf. [3, p. 83]).

Theorem 2.2. *Let x_k, $k = 1, \ldots, n$, be the nodes of the quadrature rule (1.1). Then they satisfy the following system of equations:*

$$\int_{-1}^{1} \omega_\nu(x) \sin \zeta(x - x_\nu)\, \mathrm{d}x = 0, \quad \nu = 1, \ldots, n. \tag{2.3}$$

Suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is a solution of the system of equations (2.3); under the assumption $x_k \neq x_j$, $k \neq j$, $k, j = 1, \ldots, n$, we have that x_k, $k = 1, \ldots, n$, are the nodes of the quadrature rule (1.1).

Proof. Put the function $\omega_\nu(x) \sin \zeta(x - x_\nu)$ into the quadrature rule (1.1), where x_k, $k = 1, \ldots, n$, are the nodes of the quadrature rule (1.1), and note that our quadrature rule is exact since ω_ν is a polynomial of degree $n - 1$. Therefore, we get the system of equations (2.3).

To the contrary, suppose we have some solution $\mathbf{x} = (x_1, \ldots, x_n)$ of the system (2.3). We can express functions from $\mathcal{F}_{2n}(\zeta)$ using the functions $\omega_\nu(x) \sin \zeta(x - x_\nu)$ and $\omega_\nu(x) \cos \zeta(x - x_\nu)$, $\nu = 1, \ldots, n$. Really, for $\nu = 1, \ldots, n$ we have

$$\begin{aligned}
\omega_\nu(x) \sin \zeta x &= \omega_\nu(x) \cos \zeta x_\nu \sin \zeta(x - x_\nu) + \omega_\nu(x) \sin \zeta x_\nu \cos \zeta(x - x_\nu), \\
\omega_\nu(x) \cos \zeta x &= -\omega_\nu(x) \sin \zeta x_\nu \sin \zeta(x - x_\nu) + \omega_\nu(x) \cos \zeta x_\nu \cos \zeta(x - x_\nu).
\end{aligned}$$

According to the fact that $x_k \neq x_j$, for $k \neq j$, $k, j = 1, \ldots, n$, we know that ω_ν, $\nu = 1, \ldots, n - 1$, is a basis for the linear space of polynomials of degree at most $n - 1$.

Now, all we need to do is to prove that we can solve for the weights σ_k, $k = 1, \dots, n$, of the quadrature rule (1.1), such that the formula integrates exactly the functions $\omega_\nu \cos \zeta(x - x_\nu)$, $\nu = 1, \dots, n$. This problem is solved in the proof of Theorem 2.1, and therefore there is a unique solution for the weights σ_k, $k = 1, \dots, n$. $\square$

The system of nonlinear equations (2.3) which appears in the previous theorem is the main topic for the rest of this work. Unfortunately, there are a number of solutions that solve the system (2.3), but that are not nodes of the quadrature formula (1.1). Of course those solutions of the system (2.3) are the ones which have the property $x_k = x_j$, for some $k \neq j$, $k, j = 1, \dots, n$. For example, if n is odd, there is always one trivial multiple solution of the system (2.3), namely $x_\nu = 0$, $\nu = 1, \dots, n$.

Since we are interested only in solutions which are nodes of the quadrature rule (1.1), it is important to know the behavior of the Jacobian of the system (2.3) in any such solution. We have the following result:

Theorem 2.3. *Let x_ν, $\nu = 1, \dots, n$, be the nodes of the quadrature rule (1.1); then, provided $\sigma_\nu \neq 0$, $\nu = 1, \dots, n$, the Jacobian of the solution x_ν, $\nu = 1, \dots, n$, of the system (2.3) is non-singular.*

For the proof of this theorem, we need a slight modification of the Bochner theorem [4, p. 290]. For our purpose we need a definition of a strictly positive definite function. We say the function $f : \mathbb{R} \mapsto \mathbb{C}$ is strictly positive definite provided that for every $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and every set x_k, $k = 1, \dots, n$, of real mutually different points, we have

$$\sum_{k,\nu=1}^{n} c_k \overline{c}_\nu f(x_k - x_\nu) > 0. \tag{2.4}$$

Theorem 2.4. *Let μ be a finite positive Borel measure with a bounded real support, which is symmetric with respect to zero, and which has at least one accumulation point on the real line. The function $f(x) = \int \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}\mu(t)$, is real, even and strictly positive definite.*

Proof. It is clear that f is real, since the support of the measure μ is symmetric with respect to the zero, i.e., $\int \sin xt\,\mathrm{d}\mu(t) = 0$. Also, f is even since $f(-x) = \int \mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}\mu(t) = \overline{f(x)} = f(x)$. We apply now some elementary transformations to obtain

$$\sum_{k,\nu=1}^{n} c_k \overline{c}_\nu f(x_k - x_\nu) = \sum_{k,\nu=1}^{n} c_k \overline{c}_\nu \int \mathrm{e}^{\mathrm{i}(x_k - x_\nu)t}\,\mathrm{d}\mu(t) = \int \left| \sum_{k=1}^{n} c_k \mathrm{e}^{\mathrm{i}x_k t} \right|^2 \mathrm{d}\mu(t).$$

Now suppose we have some $\mathbf{c} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, such that (2.4) does not hold. Then using the previous computation we get $\int \left|\sum_{k=1}^{n} c_k \mathrm{e}^{\mathrm{i}x_k t}\right|^2 \mathrm{d}\mu(t) = 0$, since μ is a Borel measure, and the integrand is continuous, it must be (see [3, p. 71]) the case that $\left|\sum_{k=1}^{n} c_k \mathrm{e}^{\mathrm{i}x_k t}\right|^2 = 0$, $t \in \operatorname{supp}(\mu)$, or equivalently $\sum_{k=1}^{n} c_k \mathrm{e}^{\mathrm{i}x_k t} = 0$, $t \in \operatorname{supp}(\mu)$.

According to the fact that this function is an entire function in $\mathbb{C}$, and that the support of μ has at least one accumulation point on the real line, using the standard connectedness argument, we conclude that the previous argument holds everywhere in $\mathbb{C}$, i.e., $\sum_{k=1}^{n} c_k \mathrm{e}^{\mathrm{i}x_k t} = 0$, $t \in \mathbb{C}$.

We suppose that the points x_k, $k = 1, \dots, n$, are ordered such that $x_k < x_{k+1}$, $k = 1, \dots, n-1$; if they are not we can reorder them. Further, we multiply the previous equation by $\mathrm{e}^{-\mathrm{i}x_n t}$ in order to get $\sum_{k=1}^{n-1} c_k \mathrm{e}^{\mathrm{i}(x_k - x_n)t} + c_n = 0$, $t \in \mathbb{C}$. Now, choose $t = -\mathrm{i}u$, and let u tend to $+\infty$, to obtain $c_n = 0$. Use the same procedure repeatedly to obtain $c_k = 0$, $k = 1, \dots, n$, which is a contradiction. $\square$

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. We can evaluate the Jacobian of (2.1) at any given solution in the following form:

$$\partial_{x_k} \Phi_\nu(\mathbf{x}) = -\int_{-1}^{1} \omega_{\nu,k}(x) \sin \zeta(x - x_\nu)\,\mathrm{d}x, \quad k \neq \nu, k, \nu = 1, \dots, n,$$

and

$$\partial_{x_\nu} \Phi_\nu(\mathbf{x}) = -\zeta \int_{-1}^{1} \omega_\nu(x) \cos \zeta(x - x_\nu)\,\mathrm{d}x, \quad \nu = 1, \dots, n,$$

where we exchanged the order of the differential and integral operator according to the Lebesgue theorem of dominated convergence (see [3, p. 85]). Since the polynomials $\omega_{\nu,k}$ and ω_ν are of degree at most $n-1$, we can apply the quadrature rule (1.1) to calculate the integrals, so that we get

$$\partial_{x_k} \Phi_\nu(\mathbf{x}) = \begin{cases} -\sigma_k \omega_{\nu,k}(x_k) \sin\zeta(x_k - x_\nu), & k \neq \nu, \\ -\zeta\sigma_\nu\omega_\nu(x_\nu), & k = \nu. \end{cases}$$

We can express these in the following unified form: $\partial_{x_k} \Phi_\nu(\mathbf{x}) = -\sigma_k\omega_k(x_k)\sin\zeta(x_k - x_\nu)/(x_k - x_\nu)$, $k, \nu = 1, \ldots, n$, where for $k = \nu$, we have by the continuity argument $\frac{\sin\zeta(x_k - x_\nu)}{x_k - x_\nu} \to \zeta$. This consideration gives

$$\left|\partial_{x_k} \Phi_\nu(\mathbf{x})\right|_{\nu,k=1}^n = \left(\prod_{k=1}^n \sigma_k\omega_k(x_k)\right)\left|\frac{\sin\zeta(x_k - x_\nu)}{x_k - x_\nu}\right|_{\nu,k=1}^n .$$

Using Theorem 2.4, we conclude that the function $\sin\zeta x/x$ in x is strictly positive definite since it is a Fourier transform of the Legendre measure, i.e., $\int_{-1}^1 \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t = 2\sin x/x$. This means that the matrix with elements $\sin(\zeta(x_k - x_\nu))/(x_k - x_\nu)$ cannot have zero as its eigenvalue, i.e., its determinant cannot be zero. According to the fact that we assume $\sigma_k \neq 0$, $k = 1, \ldots, n$, we have that the determinant of the Jacobian at the solution is not zero. □

The condition of this theorem, $\sigma_k \neq 0$, $k = 1, \ldots, n$, is rather natural. Assuming the contrary, i.e., $\sigma_\mu = 0$, for some $\mu = 1, \ldots, n$, produces a quadrature rule which does not depend on x_μ at all, i.e., we can choose arbitrary x_μ with the exception of the points $x_1, \ldots, x_{\mu-1}, x_{\mu+1}, \ldots, x_n$. To see this we need the following lemma.

Lemma 2.1. *Let x_k, $k = 1, \ldots, n$, be mutually different real numbers. Then x_k, $k = 1, \ldots, n$, are the nodes of the quadrature rule (1.1), with $\sigma_\mu = 0$, for some $\mu = 1, \ldots, n$, if and only if*

$$\int_{-1}^1 \omega_\mu(x)\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x = 0, \tag{2.5}$$

and

$$\int_{-1}^1 \omega_{\mu,\nu}(x)\sin\zeta(x - x_\nu)\,\mathrm{d}x = 0, \quad \nu = 1, \ldots, \mu-1, \mu+1, \ldots, n. \tag{2.6}$$

Proof. If x_k, $k = 1, \ldots, n$, are the nodes of the quadrature rule (1.1), with $\sigma_\mu = 0$, according to (2.2) and (2.3), we have (2.5). Now, choose some $\nu \neq \mu$. If we apply the quadrature formula (1.1) to the integral in (2.6) we get what is stated.

Now suppose we are given the set of mutually different points x_k, $k = 1, \ldots, n$, satisfying the properties (2.5) and (2.6). Multiply (2.5) with $\mathrm{e}^{-\mathrm{i}\zeta x_\mu}$ and take the imaginary part to get $\int_{-1}^1 \omega_\mu(x)\sin\zeta(x - x_\mu)\,\mathrm{d}x = 0$.

Choose some $\nu \neq \mu$, and multiply (2.5) by $\mathrm{e}^{-\mathrm{i}\zeta x_\nu}$ and then take its imaginary part. From that quantity subtract (2.6), multiplied by $x_\mu - x_\nu$, to get $\int_{-1}^1 \omega_\nu(x)\sin\zeta(x - x_\nu)\,\mathrm{d}x = 0$.

According to Theorem 2.2, we have the nodes of the quadrature rule (1.1). Now, multiply (2.5) by $\mathrm{e}^{-\mathrm{i}\zeta x_\mu}$ and take the real part to get $\sigma_\mu = 0$. □

A consequence of this lemma is also that, given nodes x_k, $k = 1, \ldots, n$ of the quadrature rule (1.1), $\sigma_\mu = 0$ is equivalent to $\int_{-1}^1 \omega_\mu(x)\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x = 0$.

In order to give the existence theorem, we need also the following auxiliary results.

Theorem 2.5. *For*

$$u_{2n} = \frac{\zeta^{2n}}{(2n)!}\int_{-1}^1 x^{2n}\cos\zeta x\,\mathrm{d}x, \quad n \in \mathbb{N}_0,$$

we have

$$u_{2n+2} + u_{2n} = \frac{2\sin\zeta}{\zeta}\frac{\zeta^{2n+2}}{(2n+2)!} + \frac{2\cos\zeta}{\zeta}\frac{\zeta^{2n+1}}{(2n+1)!}, \quad n \in \mathbb{N}_0, \tag{2.7}$$

and $u_0 = 2\sin\zeta/\zeta$. *Also, for* $n \in \mathbb{N}_0$,

$$u_{2n} = \frac{2\sin\zeta}{\zeta}(-1)^n \sum_{k=0}^{n}(-1)^k \frac{\zeta^{2k}}{(2k)!} + \frac{2\cos\zeta}{\zeta}(-1)^{n-1}\sum_{k=0}^{n-1}(-1)^k \frac{\zeta^{2k+1}}{(2k+1)!}. \tag{2.8}$$

Proof. Using integration by parts twice, we obtain

$$u_{2n+2} = \frac{\zeta^{2n+2}}{(2n+2)!}\int_{-1}^{1} x^{2n+2}\cos\zeta x\,\mathrm{d}x = \frac{\zeta^{2n+2}}{(2n+2)!}\frac{2\sin\zeta}{\zeta} + \frac{\zeta^{2n+1}}{(2n+1)!}\frac{2\cos\zeta}{\zeta} - u_{2n}, \quad n \in \mathbb{N}_0.$$

The last part of this theorem can be proved inductively. It is true for $n = 0$. Provided it is true for some n, replace the expression for u_{2n} in (2.7) to get the expression for u_{2n+2}. $\square$

A similar theorem holds for the sequence of moments of the sine function.

Theorem 2.6. *We have*

$$v_{2n+1} = \frac{\zeta^{2n+1}}{(2n+1)!}\int_{-1}^{1} x^{2n+1}\sin\zeta x\,\mathrm{d}x = -\frac{\zeta^{2n+1}}{(2n+1)!}\frac{2\cos\zeta}{\zeta} + u_{2n}, \quad n \in \mathbb{N}_0.$$

That is,

$$v_{2n+1} = -\frac{2\cos\zeta}{\zeta}(-1)^n \sum_{k=0}^{n}(-1)^k \frac{\zeta^{2k+1}}{(2k+1)!} + \frac{2\sin\zeta}{\zeta}(-1)^n \sum_{k=0}^{n}(-1)^k \frac{\zeta^{2k}}{(2k)!}. \tag{2.9}$$

Proof. The proof is almost identical to the proof of Theorem 2.5, so we omit it. $\square$

As a consequence of the previous two theorems we have the following statement:

Theorem 2.7. *Suppose* $\sin 2\zeta > 0$*; then* $\operatorname{sgn} u_{2n} = \operatorname{sgn}\,\sin\zeta$ *or* $u_{2n} = 0, n \in \mathbb{N}_0$*, and if* $\sin 2\zeta < 0$*, then* $\operatorname{sgn} v_{2n+1} = \operatorname{sgn}\,\sin\zeta$ *or* $v_{2n+1} = 0, n \in \mathbb{N}_0$*. For* $\sin\zeta = 0$*, we have* $\operatorname{sgn} u_{2n} = -\operatorname{sgn} v_{2n+1} = \operatorname{sgn}\,\cos\zeta$ *or* $u_{2n} = 0$ *or* $v_{2n+1} = 0, n \in \mathbb{N}_0$*, and for* $\cos\zeta = 0$*, we have* $\operatorname{sgn} u_{2n} = \operatorname{sgn} v_{2n+1} = \operatorname{sgn}\,\sin\zeta$ *or* $u_{2n} = 0$ *or* $v_{2n+1} = 0, n \in \mathbb{N}_0$*. For* $\sin 2\zeta \ge 0$*,* u_{2n} *and* u_{2n+2} *cannot both be equal to zero for* $n \in \mathbb{N}_0$*, and for* $\sin 2\zeta \le 0$*,* v_{2n+1} *and* v_{2n+3} *cannot both be equal to zero for* $n \in \mathbb{N}_0$*.*

Proof. Using the Leibnitz theorem (see [5]) for alternating series we get

$$\sum_{k=0}^{n}(-1)^k \frac{\zeta^{2k}}{(2k)!} = \cos\zeta - \lambda(-1)^{n+1}\frac{\zeta^{2n+2}}{(2n+2)!}, \quad \text{for some } \lambda \in [0,1),$$
$$\sum_{k=0}^{n}(-1)^k \frac{\zeta^{2k+1}}{(2k+1)!} = \sin\zeta - \eta(-1)^{n+1}\frac{\zeta^{2n+3}}{(2n+3)!}, \quad \text{for some } \eta \in [0,1).$$

Using these relations we obtain the following estimate for $u_{2n}, n \in \mathbb{N}_0$, when $\sin 2\zeta > 0$:

$$\begin{aligned}\frac{u_{2n}}{2} &= \frac{\sin\zeta}{\zeta}(-1)^n\left(\cos\zeta - \lambda(-1)^{n+1}\frac{\zeta^{2n+2}}{(2n+2)!}\right) + \frac{\cos\zeta}{\zeta}(-1)^{n-1}\left(\sin\zeta - \eta(-1)^n\frac{\zeta^{2n+1}}{(2n+1)!}\right)\\ &= \lambda\frac{\zeta^{2n+1}\sin\zeta}{(2n+2)!} + \eta\frac{\zeta^{2n}\cos\zeta}{(2n+1)!} = \frac{\zeta^{2n}\operatorname{sgn}(\sin\zeta)}{(2n+1)!}\left(\frac{\lambda\zeta|\sin\zeta|}{2n+2} + \eta|\cos\zeta|\right).\end{aligned}$$

Since all other quantities are positive, the sign of u_{2n} depends only on the sign of $\sin\zeta$, with possibly $\lambda = \eta = 0$ in which case $u_{2n} = 0$. The statements on the sign of $u_{2n}, n \in \mathbb{N}_0$, for $\sin\zeta = 0$ or $\cos\zeta = 0$ are obvious.

For $\upsilon_{2n+1}, n \in \mathbb{N}_0$, when $\sin 2\zeta < 0$, using similar arguments we get

$$\upsilon_{2n+1} = -\eta \frac{2\zeta^{2n+2}\cos\zeta}{(2n+3)!} + \lambda \frac{2\zeta^{2n+1}\sin\zeta}{(2n+2)!} = \frac{2\zeta^{2n+1}\operatorname{sgn}(\sin\zeta)}{(2n+2)!}\left(\frac{\eta\zeta|\cos\zeta|}{2n+3} + \lambda|\sin\zeta|\right),$$

and again since all other quantities are positive we have that the sign of $\upsilon_{2n+1}, n \in \mathbb{N}_0$, depends only on the sign of $\sin\zeta$, with possibly $\lambda = \eta = 0$ in which case $\upsilon_{2n+1} = 0$. Again, statements on the sign of $\upsilon_{2n+1}, n \in \mathbb{N}_0$, for $\cos\zeta = 0$ or $\sin\zeta = 0$ are obvious.

The last statement in this theorem can be verified from the recurrence relation (2.7) for the sequence u, and from the similar recurrence for the sequence υ. $\square$

For a fixed ζ, let us describe the set C_n of the solutions in $x_\nu, \nu = 1, \ldots, n$, of the following equation: $\int_{-1}^{1}\left(\prod_{\nu=1}^{n}(x - x_\nu)\right)\cos\zeta x \, \mathrm{d}x = 0$.

Theorem 2.8. *The set $C_n, n \geq 2$, is closed, symmetric with respect to the origin and if $\sin 2\zeta \geq 0$, we have $C_n \cap \{\mathbf{x} \in \mathbb{R}^n \mid x_\nu > 0,\ \nu = 1, \ldots, n\} = \emptyset$.*

Proof. It can be proved trivially, by the argument that the cosine function is even, that C_n is symmetric; using the Lebesgue theorem (see [3]) of dominated convergence it can be proved trivially that it is closed.

To see the last part of the statement, choose some $x_\nu > 0, \nu = 1, \ldots, n$. First note that we can expand the polynomial under the integral in the following form: $\prod_{\nu=1}^{n}(x - x_\nu) = \sum_{\nu=0}^{n}\sigma_{n,n-\nu}x^\nu$, where $\sigma_{n,\nu}$ are the elementary symmetric functions $\sigma_{n,\nu} = (-1)^\nu \sum_{(k_1,\ldots,k_\nu)} x_{k_1} \ldots x_{k_\nu}$, and where the summation is performed over all combinations, without repetition, of length ν of numbers $1, \ldots, n$. Then, $\int_{-1}^{1}\left(\prod_{\nu=1}^{n}(x - x_\nu)\right)\cos\zeta x \, \mathrm{d}x = \sum_{\nu=0}^{[n/2]}(2\nu)!\sigma_{n,n-2\nu}u_{2\nu}/\zeta^{2\nu}$, where u_{2n} is the notation from Theorem 2.5. Since $\sin 2\zeta \geq 0$, we know that all u_{2n} have the same sign and at least one of them is not equal to zero, and also all $\sigma_{n,n-2\nu}, \nu = 0, \ldots, [n/2]$, have the same sign according to the fact that $x_\nu > 0, \nu = 1, \ldots, n$, so in total all the terms in the previous sum have the same sign. This means that the sum cannot be zero, i.e., $\mathbf{x} \notin C_n$. $\square$

Using the same arguments we can describe the set S_n of the solutions of the following equation: $\int_{-1}^{1}\left(\prod_{\nu=1}^{n}(x - x_\nu)\right)\sin\zeta x \, \mathrm{d}x = 0$.

Theorem 2.9. *The set $S_n, n \geq 3$, is closed, symmetric with respect to the origin and if $\sin 2\zeta \leq 0$ we have $S_n \cap \{\mathbf{x} \in \mathbb{R}^n \mid x_\nu > 0, \nu = 1, \ldots, n\} = \emptyset$.*

Now, we are ready to prove the following theorem.

Theorem 2.10. *In the case $\sin 2\zeta \geq 0$ for $2 \leq n < \zeta/\pi - 1/2$, the system of equations (2.3) has at least $2\binom{[\zeta/\pi - 1/2]}{n}$ solutions whose nodes are all positive or all negative.*

In the case $\sin 2\zeta \leq 0$ for $3 \leq n < \zeta/\pi - 1$, the system of equations (2.3) has at least $2\binom{[\zeta/\pi - 1]}{n}$ solutions whose nodes are all positive or all negative.

Proof. First we consider the case $\sin 2\zeta \geq 0$. We rewrite the system of equations (2.3) in the following form:

$$x_\nu = \Psi_\nu^C(\mathbf{x}) = \frac{1}{\zeta}\left(\arctan\frac{\int_{-1}^{1}\omega_\nu(x)\sin\zeta x \, \mathrm{d}x}{\int_{-1}^{1}\omega_\nu(x)\cos\zeta x \, \mathrm{d}x} + k_\nu\pi\right), \quad \nu = 1, \ldots, n, k_\nu \in \mathbb{Z}. \tag{2.10}$$

The above is meaningful only for solutions which satisfy the condition $\int_{-1}^{1}\omega_\nu(x)\cos\zeta x \, \mathrm{d}x \neq 0, \nu = 1, \ldots, n$. For $\nu = n$, according to Theorem 2.8, the set of solutions of $\int_{-1}^{1}\omega_n(x)\cos\zeta x \, \mathrm{d}x = 0$ can be described as $C_{n-1} \times \mathbb{R}$. If we define the functions $p_\nu : \mathbb{R}^n \mapsto \mathbb{R}^n, \nu = 1, \ldots, n$, in the following form: $p_\nu(x_1, \ldots, x_\nu, \ldots, x_n) = (x_1, \ldots, x_n, \ldots, x_\nu), \nu = 1, \ldots, n$, we can describe the set of solutions of $\int_{-1}^{1}\omega_\nu(x)\cos\zeta x \, \mathrm{d}x = 0, \nu = 1, \ldots, n$, as $p_\nu(C_{n-1} \times \mathbb{R}), \nu = 1, \ldots, n$. In total, the transformation holds true for all the solutions which belong to the set $\mathbb{R}^n \setminus (\cup_{\nu=1}^{n} p_\nu(C_{n-1} \times \mathbb{R}))$. According to Theorem 2.8, the set C_n has empty intersection with the set $\{\mathbf{x} \mid x_\nu > 0,\ \nu = 1, \ldots, n\}$. This means $\{\mathbf{x} \mid x_\nu > 0, \nu = 1, \ldots, n\} \subset \mathbb{R}^n \setminus \left(\cup_{\nu=1}^{n} p_\nu(C_{n-1} \times \mathbb{R})\right)$.

Table 1
Nodes x_k and weights σ_k, $k = 1, \ldots, n$, for $n = 10$, $\zeta = 1000$

k	x_k	σ_k	x_k	σ_k
1	0.08227317490466181	3.3326095191855526(1)	0.06970639005564088	1.5445670386820788(2)
2	0.2236447375670234	3.2427372948443411(2)	0.1545293231919083	7.9727221915702422(2)
3	0.3650163223412639	1.5938983691119251(3)	0.3178920342621190	− 1.8088877578920682(4)
4	0.5001047404753007	5.1833156081932871(3)	0.3493079432435246	2.6162729572152158(4)
5	0.6226268036316312	1.1842203747575649(4)	0.4498388572151283	− 1.5630764084997561(4)
6	0.7325825073981575	1.9941676453852423(4)	0.5503697780502463	1.0027815934954146(4)
7	0.8268302572329836	− 2.4825959253141296(4)	0.7043077616542470	8.6464161422821680(3)
8	0.9085116428079892	2.6321931851560798(4)	0.7577148201208147	6.6167406191733902(3)
9	0.9556355200068546	1.8123285743890243(4)	0.8519625749534823	− 1.1974650342877282(3)
10	0.9933346224203500	− 4.6474741928722418(3)	0.9493519410276697	− 1.1341982560929190(2)

Thus, any solution of the system (2.3) with all positive nodes will also be the solution of the system (2.10). According to Theorem 2.8, the set C_{n-1} is symmetric with respect to the origin. The same holds for $\cup_{\nu=1}^{n} p_\nu(C_{n-1} \times \mathbb{R})$. This means that everything proved for the quadrature formula with all positive nodes holds true for a quadrature formula with all negative nodes.

Now, choose some fixed vector, with strictly increasing coordinates, of positive integers $\mathbf{k} = (k_1, \ldots, k_n)$, with the property $k_n < \zeta/\pi - 1/2$. The functions $\Psi_\nu^C(\mathbf{x})$, $\nu = 1, \ldots, n$, defined in (2.10), are continuous in $\mathbf{x}$ for $x_\nu > 0$, $\nu = 1, \ldots, n$. Construct the mapping $\boldsymbol{\Psi}_{\mathbf{k}}^C : \mathbb{R}^n \mapsto \mathbb{R}^n$ in the following form: $\boldsymbol{\Psi}_{\mathbf{k}}^C(\mathbf{x}) = (\Psi_1^C(\mathbf{x}), \ldots, \Psi_n^C(\mathbf{x}))$. This map is continuous in $\mathbf{x}$, for $x_\nu > 0$, $\nu = 1, \ldots, n$, according to the fact that coordinate maps are continuous. The mapping $\boldsymbol{\Psi}_{\mathbf{k}}^C$ maps continuously the closed convex set $A_{\mathbf{k}} = \times_{\nu=1}^{n} \left[\left(k_\nu - \frac{1}{2}\right)\frac{\pi}{\zeta}, \left(k_\nu + \frac{1}{2}\right)\frac{\pi}{\zeta}\right]$ into itself. According to the Brouwer fixed point theorem (cf. [6, pp. 161–162]), the map $\boldsymbol{\Psi}_{\mathbf{k}}^C$ has a fixed point, i.e., there exists the point $\mathbf{x_k} \in A_{\mathbf{k}}$ which satisfies the system of equations $\mathbf{x_k} = \boldsymbol{\Psi}_{\mathbf{k}}^C(\mathbf{x_k})$. According to the fact that $\int_{-1}^{1} \omega_\nu(x) \cos \zeta x \, \mathrm{d}x \neq 0$, we conclude easily that we cannot have a solution with the ν-th coordinate equal to $(k_\nu \pm 1/2)\pi/\zeta$. This means that all coordinates of the solution $\mathbf{x_k}$ are different, according to the fact that the coordinates of the vector $\mathbf{k}$ are different. Hence, $\mathbf{x_k}$ are the nodes of the quadrature rule (1.1). According to Lemma 2.1, we know that for this solution all the weights are different from zero.

The number of the solutions can be established rather easily. Using the previous arguments we can prove that there are exactly the same number of solutions as there are ways we can choose orderless n integers from the set $\{1, \ldots, [\zeta/\pi - 1/2]\}$, multiplied by two to count the solutions with all negative coordinates.

For the case $\sin 2\zeta \leq 0$, we rewrite the system of equations (2.3) in the following form:

$$x_\nu = \Psi_\nu^S(\mathbf{x}) = \frac{1}{\zeta}\left(\operatorname{arccot}\frac{\int_{-1}^{1} \omega_\nu(x) \cos \zeta x \, \mathrm{d}x}{\int_{-1}^{1} \omega_\nu(x) \sin \zeta x \, \mathrm{d}x} + k_\nu \pi\right), \quad \nu = 1, \ldots, n, k_\nu \in \mathbb{Z}. \tag{2.11}$$

Using the same arguments we can prove that $\int_{-1}^{1} \omega_\nu(x) \sin \zeta x \, \mathrm{d}x \neq 0$, $\nu = 1, \ldots, n$, on the set $\{\mathbf{x} \mid x_\nu > 0, \nu = 1, \ldots, n\}$. Constructing the map $\boldsymbol{\Psi}_{\mathbf{k}}^S : \mathbb{R}^n \mapsto \mathbb{R}^n$, with $\boldsymbol{\Psi}_{\mathbf{k}}^S(\mathbf{x}) = (\Psi_1^S(\mathbf{x}), \ldots, \Psi_n^S(\mathbf{x}))$, we can prove that it has a fixed point in the set $B_{\mathbf{k}} = \times_{\nu=1}^{n} [k_\nu\pi/\zeta, (k_\nu + 1)\pi/\zeta]$.

Like in the previous case, we see that there are exactly the same number of solutions as there are ways we can choose orderless n integers from the set $\{1, \ldots, [\zeta/\pi - 1]\}$, multiplied by two to count the solutions with all negative coordinates. □

3. Numerical example

The nodes x_k, $k = 1, \ldots, n$, of the quadrature formula (1.1) can be obtained by an application of the Newton–Kantorovich method to the system (2.3) with appropriately chosen starting values. Theorem 2.3 guarantees that the Jacobian of the given solution is regular. Once nodes are constructed, weights σ_k, $k = 1, \ldots, n$, can be computed using formula (2.2). In Table 1 we give two different quadrature rules with all positive nodes for the case $n = 10$, $\zeta = 1000$. Numbers in parentheses indicate decimal exponents. All computations were performed using the *Mathematica* package OrthogonalPolynomials [7].

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